Chapter review 6

1 a $\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \mathrm{e}^x \cos x$

$\mathrm{e}^{\int \tan x \,\mathrm{d}x} = \sec x$

$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} + y\sec x \tan x = e^x$

$y\sec x = \mathrm{e}^x + k$

$y = \mathrm{e}^x \cos x + k\cos x$

b at $x = \pi$, $y = 1$

$\mathrm{e}^\pi \cos\pi + k\cos\pi = 1$

$k = \frac{1 - \mathrm{e}^\pi \cos\pi}{\cos\pi}$

$k = -\left(1 + \mathrm{e}^\pi\right)$

Therefore:

$y = \mathrm{e}^x \cos x - \left(1 + \mathrm{e}^\pi\right)\cos x$

$= \left(\mathrm{e}^x - \mathrm{e}^\pi - 1\right)\cos x$

2 $\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = \sin x$

$\mathrm{e}^{-3\int \mathrm{d}x} = \mathrm{e}^{-3x}$

$\mathrm{e}^{-3x} - 3\mathrm{e}^{-3x}y = \sin x$

$\mathrm{e}^{-3x}y = \int \mathrm{e}^{-3x}\sin x \,\mathrm{d}x$

$$\begin{aligned}\int \mathrm{e}^{-3x}\sin x\,\mathrm{d}x &= -\mathrm{e}^{-3x}\cos x - \int 3\mathrm{e}^{-3x}\cos x\,\mathrm{d}x \\ &= -\mathrm{e}^{-3x}\cos x - 3\left[-\mathrm{e}^{-3x}\sin x - \int 3\mathrm{e}^{-3x}\sin x\right] \\ &= -\mathrm{e}^{-3x}\cos x - 3\mathrm{e}^{-3x}\sin x - 9\int \mathrm{e}^{-3x}\sin x\,\mathrm{d}x \\ &= -\frac{1}{10}\mathrm{e}^{-3x}(3\sin x + \cos x)\end{aligned}$$

$\int \mathrm{e}^{-3x}\sin x\,\mathrm{d}x = -\mathrm{e}^{-3x}\cos x - 3\mathrm{e}^{-3x}\sin x - 9\int \mathrm{e}^{-3x}\sin x\,\mathrm{d}x$

$\int \mathrm{e}^{-3x}\sin x\,\mathrm{d}x = -\frac{1}{10}\mathrm{e}^{-3x}(3\sin x + \cos x)$

$\mathrm{e}^{-3x}y = -\frac{1}{10}\mathrm{e}^{-3x}(3\sin x + \cos x) + A$

$y = -\frac{1}{10}(3\sin x + \cos x) + A\mathrm{e}^{3x}$

At $x = 0, y = 0$

$-\frac{1}{10} + A = 0$

$A = \frac{1}{10}$

$y = -\frac{1}{10}(3\sin x + \cos x) + \frac{1}{10}\mathrm{e}^{3x}$

3 $\frac{dy}{dx} = x(4 - y^2)$

$\frac{1}{4 - y^2}\frac{dy}{dx} = x$

$\int \frac{1}{4 - y^2}\,dy = \int x\,dx$

$\frac{1}{4 - y^2} = \frac{1}{(2 - y)(2 + y)}$

$\frac{1}{(2 - y)(2 + y)} = \frac{A}{2 - y} + \frac{B}{2 + y}$

$1 = A(2 + y) + B(2 - y)$

When $y = 2$

$4A = 1$

$A = \frac{1}{4}$

When $y = -2$

$4B = 1$

$B = \frac{1}{4}$

Therefore:

$\frac{1}{(2 - y)(2 + y)} = \frac{1}{4(2 - y)} + \frac{1}{4(2 + y)}$

$\int \frac{1}{4 - y^2}\,dy = \frac{1}{4}\int \frac{1}{2 - y}\,dy + \frac{1}{4}\int \frac{1}{2 + y}\,dy$

$\frac{1}{4}\int \frac{1}{2 - y}\,dy + \frac{1}{4}\int \frac{1}{2 + y}\,dy = \int x\,dx$

$\int \frac{1}{2 - y}\,dy + \int \frac{1}{2 + y}\,dy = 4\int x\,dx$

$-\ln(2 - y) + \ln(2 + y) = 2x^2 + c$

$\ln\left(\frac{2 + y}{2 - y}\right) = 2x^2 + c$

$$\begin{aligned}\frac{2 + y}{2 - y} &= e^{2x^2 + c}\\ &= e^{2x^2}e^{c}\\ &= Ae^{2x^2}\end{aligned}$$

Let $u = Ae^{2x^2}$

$\frac{2 + y}{2 - y} = u$

$2 + y = u(2 - y)$

$2 + y = 2u - uy$

$uy + y = 2u - 2$

$y = \frac{2(Ae^{2x^2} - 1)}{Ae^{2x^2} + 1}$

When $x = 0$, $y = 1$

$$1 = \frac{2(A-1)}{A+1}$$

$$A + 1 = 2A - 2$$

$$A = 3$$

Therefore:

$$y = \frac{2\left(3e^{2x^2} - 1\right)}{3e^{2x^2} + 1}$$

4 $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

$$y = e^{-\frac{1}{2}x}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$

5 $\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 36y = 0$

$$m^2 - 12m + 36 = 0$$

$$(m-6)(m-6) = 0$$

$$m = 6$$

$$y = (A + Bx)e^{6x}$$

6 $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 0$

$$m^2 - 4m = 0$$

$$m(m-4) = 0$$

$$m = 0 \text{ or } m = 4$$

$$y = A + Be^{4x}$$

7 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+k^2y=0$

$m^2+k^2=0$

$m^2=-k^2$

$m=\pm k\mathrm{i}$

$y=A\cos kx+B\sin kx$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-kA\sin kx+kB\cos kx$

When $x=0$, $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=1$

$A=1$

$kB=1$

$B=\frac{1}{k}$

$y=\cos kx+\frac{1}{k}\sin kx$

8 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+10y=0$

$m^2-2m+10=0$

$m=\frac{2\pm\sqrt{4-40}}{2}$

$=\frac{2\pm\sqrt{-36}}{2}$

$=1\pm3\mathrm{i}$

$y=\mathrm{e}^x\left(A\cos3x+B\sin3x\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x\left(A\cos3x+B\sin3x\right)+\mathrm{e}^x\left(-3A\sin3x+3B\cos3x\right)$

When $x=0$, $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=3$

$A=0$

$A+3B=3$

$B=1$

$y=\mathrm{e}^x\sin3x$

9 a $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}$ **(1)**

Let $y = ke^{2x}$

$\frac{dy}{dx} = 2ke^{2x}$

$\frac{d^2y}{dx^2} = 4ke^{2x}$

Substituting into **(1)** gives:

$4ke^{2x} - 4(4ke^{2x}) + 13ke^{2x} = e^{2x}$

$k = 1$

Hence the particular integral is e^{2x}

b $m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2}$

$= \frac{4 \pm \sqrt{-36}}{2}$

$= 2 \pm 3i$

Therefore the complementary function is:

$y = e^{2x}(A\cos 3x + B\sin 3x)$

And the general solution is:

$y = e^{2x}(A\cos 3x + B\sin 3x) + e^{2x}$

10 $\frac{d^2y}{dx^2} - y = 4e^x$ **(1)**

Let $y = Axe^x$

$\frac{dy}{dx} = Axe^x + Ae^x$

$\frac{d^2y}{dx^2} = Axe^x + 2Ae^x$

Substituting into **(1)** gives:

$Axe^x + 2Ae^x - Axe^x = 4e^x$

$2A = 4$

$A = 2$

Hence the particular integral is $2xe^x$

$m^2 - 1 = 0$

$m = \pm 1$

Therefore the complementary function is:

$y = Ae^x + Be^{-x}$

And the general solution is:

$y = Ae^x + Be^{-x} + 2xe^x$

11 a $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ **(1)**

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

Therefore the complementary function is:

$y = (A + Bx)e^{2x}$

b Let $y = \lambda e^{2x}$

$\frac{dy}{dx} = 2\lambda e^{2x}$

$\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

Substituting into **(1)** gives:

$4\lambda e^{2x} - 8\lambda e^{2x} + 4\lambda e^{2x} = 4e^{2x}$

$0 = 4e^{2x}$

This is not possible, therefore λe^{2x} cannot be the particular integral.

Let $y = \lambda x e^{2x}$

$\frac{dy}{dx} = 2\lambda x e^{2x} + \lambda e^{2x}$

$\frac{d^2y}{dx^2} = 4\lambda x e^{2x} + 2\lambda e^{2x} + 2\lambda e^{2x}$

$= 4\lambda x e^{2x} + 4\lambda e^{2x}$

Substituting into **(1)** gives:

$4\lambda x e^{2x} + 4\lambda e^{2x} - 4(2\lambda x e^{2x} + \lambda e^{2x}) + 4\lambda x e^{2x} = 4e^{2x}$

$4\lambda x e^{2x} + 4\lambda e^{2x} - 8\lambda x e^{2x} - 4\lambda e^{2x} + 4\lambda x e^{2x} = 4e^{2x}$

$0 = 4e^{2x}$

This is not possible, therefore $\lambda x e^{2x}$ cannot be the particular integral.

c Let $y = kx^2e^{2x}$

$\frac{dy}{dx} = 2kx^2e^{2x} + 2kxe^{2x}$

$\frac{d^2y}{dx^2} = 4kx^2e^{2x} + 8kxe^{2x} + 2ke^{2x}$

Substituting into **(1)** gives:

$4kx^2e^{2x} + 8kxe^{2x} + 2ke^{2x} - 4(2kx^2e^{2x} + 2kxe^{2x}) + 4kx^2e^{2x} = 4e^{2x}$

$4kx^2e^{2x} + 8kxe^{2x} + 2ke^{2x} - 8kx^2e^{2x} - 8kxe^{2x} + 4kx^2e^{2x} = 4e^{2x}$

Comparing coefficients for constant terms:

$2k = 4$

$k = 2$

Hence the particular integral is $2x^2e^{2x}$

~~A~~and the general solution is:

$y = (A + Bx)e^{2x} + 2x^2e^{2x}$

$= (A + Bx + 2x^2)e^{2x}$

12 $\frac{d^2y}{dt^2}+4y=5\cos 3t$ **(1)**

Let $y=A\cos 3t+B\sin 3t$

$\frac{dy}{dt}=-3A\sin 3t+3B\cos 3t$

$\frac{d^2y}{dt^2}=-9A\cos 3t-9B\sin 3t$

Substituting into **(1)** gives:

$-9A\cos 3t-9B\sin 3t+4\left(A\cos 3t+B\sin 3t\right)=5\cos 3t$

$-9A\cos 3t-9B\sin 3t+4A\cos 3t+4B\sin 3t=5\cos 3t$

$-5A\cos 3t-5B\sin 3t=5\cos 3t$

Comparing coefficients:

For cos $3t$:

$-5A=5$

$A=-1$

For sin $3t$:

$-5B=0$

$B=0$

Hence the particular integral is $-\cos 3t$

$m^2+4=0$

$m=\pm 2\text{i}$

Therefore the complementary function is:

$y=A\cos 2t+B\sin 2t$

And the general solution is:

$y=A\cos 2t+B\sin 2t-\cos 3t$

$\frac{dy}{dt}=-2A\sin 2t+2B\cos 2t+3\sin 3t$

When $t=0$, $y=1$ and $\frac{dy}{dt}=2$

$A=1$

$B=1$

Therefore the particular solution is:

$y=\cos 2t+\sin 2t-\cos 3t$

13 a $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x}$ **(1)**

Let $y = \lambda + \mu x + kxe^{2x}$

$\frac{dy}{dx} = \mu + 2kxe^{2x} + ke^{2x}$

$\frac{d^2y}{dx^2} = 4kxe^{2x} + 2ke^{2x} + 2ke^{2x}$

$= 4kxe^{2x} + 4ke^{2x}$

Substituting into (1) gives:

$4kxe^{2x} + 4ke^{2x} - 3(\mu + 2kxe^{2x} + ke^{2x}) + 2(\lambda + \mu x + kxe^{2x}) = 4x + e^{2x}$

$4kxe^{2x} + 4ke^{2x} - 3\mu - 6kxe^{2x} - 3ke^{2x} + 2\lambda + 2\mu x + 2kxe^{2x} = 4x + e^{2x}$

$ke^{2x} - 3\mu + 2\lambda + 2\mu x = 4x + e^{2x}$

Comparing coefficients:

For e^{2x}:

$k = 1$

For x:

$2\mu = 4$

$\mu = 2$

For constant terms:

$-3\mu + 2\lambda = 0$

$-6 + 2\lambda = 0$

$\lambda = 3$

Hence the particular integral is $3 + 2x + xe^{2x}$

b $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1$ or $m = 2$

Therefore the complementary function is:

$y = Ae^{2x} + Be^{x}$

And the general solution is:

$y = Ae^{2x} + Be^{x} + xe^{2x} + 2x + 3$

14 a $16\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}+5y=5x+23$ **(1)**

Let $y=Ax+B$

$\dfrac{dy}{dx}=A$

$\dfrac{d^2y}{dx^2}=0$

Substituting into **(1)** gives:

$8A+5(Ax+B)=5x+23$

$8A+5Ax+5B=5x+23$

Comparing coefficients:

For x:

$5A=5\Rightarrow A=1$

~~$A=1$~~

For constant terms:

$8A+5B=23\Rightarrow B=3$

~~$B=3$~~

Hence the particular integral is $x+3$

$16m^2+8m+5=0$

$$m=\frac{-8\pm\sqrt{8^2-4(16)(5)}}{2(16)}$$

$$=\frac{-8\pm\sqrt{-256}}{32}$$

$$=\frac{-8\pm16\text{i}}{32}$$

$$=-\frac{1}{4}\pm\frac{1}{2}\text{i}$$

Therefore the complementary function is:

$$y=e^{-\frac{1}{4}x}\left(A\cos\left(\frac{1}{2}x\right)+B\sin\left(\frac{1}{2}x\right)\right)$$

And the general solution is:

$$y=e^{-\frac{1}{4}x}\left(A\cos\left(\frac{1}{2}x\right)+B\sin\left(\frac{1}{2}x\right)\right)+x+3$$

$$\frac{dy}{dx}=e^{-\frac{1}{4}x}\left(-\frac{1}{2}A\sin\left(\frac{1}{2}x\right)+\frac{1}{2}B\cos\left(\frac{1}{2}x\right)\right)-\frac{1}{4}e^{-\frac{1}{4}x}\left(A\cos\left(\frac{1}{2}x\right)+B\sin\left(\frac{1}{2}x\right)\right)+1$$

When $x=0$, $y=3$ and $\dfrac{dy}{dx}=3$

$A+3=3$

$A=0$

$\dfrac{1}{2}B-\dfrac{1}{4}A+1=3$

$B=4$

Therefore the particular solution is:

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$y = 4e^{-\frac{1}{4}x}\sin\left(\frac{1}{2}x\right) + x + 3$

b As $x \to \infty$, $4e^{-\frac{1}{4}x}\sin\left(\frac{1}{2}x\right) \to 0$ so $y \to x + 3$

15 $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\sin 3x - 2\cos 3x$ **(1)**

Let $A\cos 3x + B\sin 3x$

$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x$

$\frac{d^2y}{dx^2} = -9A\cos 3x - 9B\sin 3x$

Substituting into **(1)** gives:

$-9A\cos 3x - 9B\sin 3x + 3A\sin 3x - 3B\cos 3x - 6A\cos 3x - 6B\sin 3x = 3\sin 3x - 2\cos 3x$

$-15A\cos 3x - 15B\sin 3x + 3A\sin 3x - 3B\cos 3x = 3\sin 3x - 2\cos 3x$

$\cos 3x(-15A - 3B) + \sin 3x(3A - 15B) = 3\sin 3x - 2\cos 3x$

Comparing coefficients:

For cos 3x:

$-15A - 3B = -2$ **(1)**

For sin 3x:

$3A - 15B = 3$ **(2)**

Adding **(1)** and 5 × **(2)** gives:

$-78B = 13$

$B = -\frac{1}{6}$

$A = \frac{1}{6}$

Hence the particular integral is $\frac{1}{6}\cos 3x - \frac{1}{6}\sin 3x$

$m^2 - m - 6 = 0$

$(m + 2)(m - 3) = 0$

$m = -2$ or $m = 3$

Therefore the complementary function is:

$y = Ae^{3x} + Be^{-2x}$

And the general solution is:

$y = Ae^{3x} + Be^{-2x} + \frac{1}{6}\cos 3x - \frac{1}{6}\sin 3x$

If $y(x)$ remains finite as $x \to \infty$ then $A = 0$

Therefore:

$y = Be^{-2x} + \frac{1}{6}\cos 3x - \frac{1}{6}\sin 3x$

When $x = 0$, $y = 1$

$1 = B + \frac{1}{6}$

$B = \frac{5}{6}$

Therefore the particular solution is:

$y = \frac{1}{6}\left(5e^{-2x} + \cos 3x - \sin 3x\right)$

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16 a $\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=\cos 4t$, $t\geq 0$ **(1)**

Let $x=A\cos 4t+B\sin 4t$

$\frac{dx}{dt}=-4A\sin 4t+4B\cos 4t$

$\frac{d^2x}{dt^2}=-16A\cos 4t-16B\sin 4t$

Substituting into **(1)** gives:

$-16A\cos 4t-16B\sin 4t+8(-4A\sin 4t+4B\cos 4t)+16(A\cos 4t+B\sin 4t)=\cos 4t$

$-16A\cos 4t-16B\sin 4t-32A\sin 4t+32B\cos 4t+16A\cos 4t+16B\sin 4t=\cos 4t$

$-32A\sin 4t+32B\cos 4t=\cos 4t$

Comparing coefficients:

For cos 4t:

$32B=1$

$B=\frac{1}{32}$

For sin 4t:

$-32A=0$

$A=0$

Hence the particular integral is $\frac{1}{32}\sin 4t$

$m^2+8m+16=0$

$(m+4)(m+4)=0$

$m=-4$

Therefore the complementary function is:

$x=(A+Bt)e^{-4t}$

And the general solution is:

$x=(A+Bt)e^{-4t}+\frac{1}{32}\sin 4t$

16 b $x = (A + Bt)e^{-4t} + \frac{1}{32}\sin 4t$

$\frac{dx}{dt} = -4(A + Bt)e^{-4t} + Be^{-4t} + \frac{1}{8}\cos 4t$

When $t = 0$, $x = \frac{1}{2}$ and $\frac{dy}{dt} = 0$

$A = \frac{1}{2}$

$-4A + B + \frac{1}{8} = 0$

$B = \frac{15}{8}$

Therefore the particular solution is:

$x = \frac{1}{8}(4 + 15t)e^{-4t} + \frac{1}{32}\sin 4t$

c As $t \to \infty$ the e^{-4t} dominates the first term so $\frac{1}{8}(4 + 15t)e^{-4t} \to 0$ leaving:

$x = \frac{1}{32}\sin 4t$ which is an oscillation.

17 a Let $x = e^u$, then $\frac{dx}{du} = e^u$

and $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx} = x\frac{dy}{dx}$

$$\frac{d^2y}{du^2} = \frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du}$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

$\therefore \quad x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x \Rightarrow \frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = \ln x = u \quad *$

> Find $\frac{dy}{du}$ in terms of x and $\frac{dy}{dx}$, and show that $\frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$ then substitute into the differential equation.

The auxiliary equation is

$m^2 + 3m + 2 = 0$

$\therefore \quad (m+2)(m+1) = 0$

$\Rightarrow \quad m = -1 \text{ or } -2$

$\therefore$ The c.f. is $y = Ae^{-u} + Be^{-2u}$

Let the p.i. be $y = \lambda u + \mu \Rightarrow \frac{dy}{du} = \lambda,\ \frac{d^2y}{du^2} = 0$

Substitute into $*$

$\therefore\ 3\lambda + 2\lambda u + 2\mu = u$

Equate coefficients of u: $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

constants: $3\lambda + 2\mu = 0 \quad \therefore \quad \mu = -\frac{3}{4}$

$\therefore$ The p.i. is $y = \frac{1}{2}u - \frac{3}{4}$

The general solution is $y = Ae^{-u} + Be^{-2u} + \frac{1}{2}u - \frac{3}{4}$

But $x = e^u \to u = \ln x$

Also $e^{-u} = x^{-1} = \frac{1}{x}$ and $e^{-2u} = x^{-2} = \frac{1}{x^2}$

$\therefore$ The general solution of the original equation is $y = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{2}\ln x - \frac{3}{4}$

17 b But $y=1$ when $x=1$

$\therefore\ 1=A+B-\frac{3}{4}\Rightarrow A+B=\frac{7}{4}$ **(1)**

$$\frac{dy}{dx}=-\frac{A}{x^2}-\frac{2B}{x^3}+\frac{1}{2x}$$

When $x=1$, $\frac{dy}{dx}=1$

$\therefore\ 1=-A-2B+\frac{1}{2}\Rightarrow A+2B=-\frac{1}{2}$ **(2)**

Solve the simultaneous equations **(1)** and **(2)** to give $B=-\frac{9}{4}$ and $A=4$

$\therefore$ The equation of the solution curve described is $y=\frac{4}{x}-\frac{9}{4x^2}+\frac{1}{2}\ln x-\frac{3}{4}$

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18 $z = \sin x \quad \therefore \quad \frac{dz}{dx} = \cos x$ and $\frac{dy}{dx} = \frac{dy}{dz} \times \cos x$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{dy}{dz}\sin x + \cos x \frac{d^2y}{dz^2} \times \frac{dz}{dx}$$

$$= -\frac{dy}{dz}\sin x + \cos^2 x \frac{d^2y}{dz^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = \cos^2 x \, e^{\sin x} \quad \dagger$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz} + \tan x \cos x \frac{dy}{dz} + y\cos^2 x = \cos^2 x \, e^z$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = e^z \quad *$$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm\, i$

$\therefore$ The c.f. is $y = A\cos z + B\sin z$

The p.i. is $y = \lambda e^z \Rightarrow \frac{dy}{dz} = \lambda e^z$ and $\frac{d^2y}{dz^2} = \lambda e^z$

Substitute in $*$ to give

$2\lambda e^z = e^z \Rightarrow \lambda = \frac{1}{2}$

$\therefore$ The general solution of $*$ is $y = A\cos z + B\sin z + \frac{1}{2}e^z$

The original equation $\dagger$ has solution

$y = A\cos(\sin x) + B\sin(\sin x) + \frac{1}{2}e^{\sin x}$

But $y = 1$ when $x = 0$

$\therefore \quad 1 = A + \frac{1}{2} \Rightarrow A = \frac{1}{2}$

$\frac{dy}{dx} = \cos x(-A\sin(\sin x)) + \cos x(B\cos(\sin x)) + \frac{1}{2}\cos x \, e^{\sin x}$

As $\frac{dy}{dx} = 3$ when $x = 0$

$\therefore \quad 3 = B + \frac{1}{2} \Rightarrow B = \frac{5}{2}$

$\therefore \quad y = \frac{1}{2}\cos(\sin x) + \frac{5}{2}\sin(\sin x) + \frac{1}{2}e^{\sin x}$

Challenge

1 a Given that $z = y^2$, and so $y = z^{\frac{1}{2}}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}z^{-\frac{1}{2}}\frac{\mathrm{d}z}{\mathrm{d}x}$

The equation $2(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \frac{1}{y}$ becomes

$$2(1+x^2)\times\frac{1}{2}z^{-\frac{1}{2}}\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz^{\frac{1}{2}} = z^{-\frac{1}{2}}$$

Multiply the equation by $\frac{z^{\frac{1}{2}}}{1+x^2}$

Then $\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{2x}{1+x^2}z = \frac{1}{1+x^2}$

The integrating factor is $\mathrm{e}^{\int\frac{2x}{1+x^2}\mathrm{d}x} = \mathrm{e}^{\ln(1+x^2)} = 1+x^2$

$$\therefore\ (1+x^2)\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 1$$

$$\therefore\ \frac{\mathrm{d}}{\mathrm{d}x}[(1+x^2)z] = 1$$

$$\therefore\ (1+x^2)z = \int 1\,\mathrm{d}x$$

$$= x + c$$

$$\therefore\ z = \frac{x+c}{(1+x^2)}$$

As $y = z^{\frac{1}{2}}$, $\quad y = \sqrt{\frac{x+c}{(1+x^2)}}$

b When $x = 0,\ y = 2 \qquad \therefore 2 = \sqrt{c} \Rightarrow c = 4$

$$\therefore y = \sqrt{\frac{x+4}{1+x^2}}$$

2 a $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right)$$
$$= \frac{dy}{du}\frac{d^2u}{dx^2} + \frac{du}{dx}\left(\frac{d^2y}{du^2}\frac{du}{dx}\right)$$
$$= \frac{dy}{du}\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2\frac{d^2y}{du^2}$$

Let $x = e^u$, therefore:

$u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$
$$= e^{-u}$$

$$\frac{d^2u}{dx^2} = -x^{-2}$$
$$= -e^{-2u}$$

The original equation is:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x$$

The transformed equation is:

$$e^{2u}\left[\frac{dy}{du}\left(-e^{-2u}\right) + e^{-2u}\frac{d^2y}{du^2}\right] + 4e^u\left[\frac{dy}{du}e^{-u}\right] + 2y = u$$

$$-\frac{dy}{du} + \frac{d^2y}{du^2} + 4\frac{dy}{du} + 2y = u$$

$$\frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = u \qquad \textbf{(1)}$$

Let $y = Au + B$

$$\frac{dy}{du} = A$$

$$\frac{d^2y}{du^2} = 0$$

Substituting into **(1)** gives:

$3A + 2(Au + B) = u$

Comparing coefficients:

For u:

$2A = 1$

$$A = \frac{1}{2}$$

For constant terms:

$3A + 2B = 0$

$$\frac{3}{2} + 2B = 0$$

$$B = -\frac{3}{4}$$

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Hence the particular integral is $\frac{1}{2}u-\frac{3}{4}$

$\frac{d^2y}{du^2}+3\frac{dy}{du}+2y=u$

$m^2+3m+2=0$

$(m+1)(m+2)=0$

$m=-1$ or $m=-2$

Therefore the complementary function is:

$y=Ae^{-u}+Be^{-2u}$

And the general solution is:

$y=Ae^{-u}+Be^{-2u}+\frac{1}{2}u-\frac{3}{4}$

Therefore:

$y=\frac{A}{x}+\frac{B}{x^2}+\frac{1}{2}\ln x-\frac{3}{4}$

2 b $y=\frac{A}{x}+\frac{B}{x^2}+\frac{1}{2}\ln x-\frac{3}{4}$

$\frac{dy}{dx}=-\frac{A}{x^2}-\frac{2B}{x^3}+\frac{1}{2x}$

When $x=1$, $y=1$ and $\frac{dy}{dx}=1$

$A+B-\frac{3}{4}=1$

$A+B=\frac{7}{4}$ **(2)**

$-A-2B+\frac{1}{2}=1$

$-A-2B=\frac{1}{2}$ **(3)**

Adding **(2)** and **(3)** gives:

$B=-\frac{9}{4}$

$A+B=\frac{7}{4}$

$A-\frac{9}{4}=\frac{7}{4}$

$A=4$

Therefore the particular solution is:

$y=\frac{4}{x}+-\frac{9}{4x^2}+\frac{1}{2}\ln x-\frac{3}{4}$

3 Substitute $u=\frac{dy}{dx}$ so equation becomes

$\frac{du}{dx} = u^2$

$\Rightarrow \int \frac{du}{u^2} = \int dx$

$\Rightarrow -\frac{1}{u} = x + B$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{x+B}$

$\Rightarrow y = -\ln(x+B) + A$

$= A - \ln(x+B)$ as required.

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